A new formula for the energy functionals E_k and its applications

Haozhao Li

February 2, 2008

Abstract

We give a new formula for the energy functionals E_k defined by Chen-Tian [5], and discuss the relations between these functionals. We also apply our formula to give a new proof of the fact that the holomorphic invariants corresponding to the E_k functionals are equal to the Futaki invariant.

1 Introduction

In [5], a series of energy functionals $E_k(k = 0, 1, \cdots, n)$ were introduced by X.X. Chen and G. Tian which were used to prove the convergence of the Kähler Ricci flow under some curvature assumptions. The first energy functional E_0 of this series is exactly the K-energy introduced by Mabuchi in [12], which can be defined for any Kähler potential $\varphi(t)$ on a Kähler manifold (M, ω) as follows:

$$\frac{d}{dt}E_0(\varphi(t)) = -\frac{1}{V}\int_M \frac{\partial\varphi}{\partial t}(R_\varphi - r)\omega_\varphi^n.$$

Here R_φ is the scalar curvature with respect to the Kähler metric $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$, $r = \frac{[c_1(M)][\omega]^{n-1}}{[\omega]^n}$ is the average of R_φ and $V = [\omega]^n$ is the volume.

It is well-known that the behavior of the K-energy plays a central role on the existence of Kähler-Einstein metrics and constant scalar curvature metrics. In [1], Bando-Mabuchi proved that the K-energy is bounded from below on a Kähler-Einstein manifold with $c_1(M) > 0$. It has been shown by G. Tian in [16][17] that M admits a Kähler-Einstein metric if and only if the K-energy is proper. Recently, Chen-Tian in [7] extended these results to extremal Kähler metrics, and Cao-Tian-Zhu in [2][18] proved similar results on Kähler Ricci solitons. So a natural question is how the energy functionals E_k are related to these extremal metrics.

Following a question posed by Chen in [3], Song-Weinkove recently proved in [14] that the energy functionals E_k have a lower bound on the space of Kähler metrics with nonnegative Ricci curvature for Kähler-Einstein manifolds. Moreover, they also showed that modulo holomorphic vector fields, E_1 is proper if and only if there exists a Kähler-Einstein metric. Shortly afterwards, N. Pali [13] gave a formula between E_1 and the K-energy E_0, which implies E_1 has a lower bound if the K-energy is bounded from below. Tosatti [19] proved under some curvature assumptions, the critical point of E_k is a Kähler-Einstein metric. Pali's theorem says that the functional E_1 is always bigger than the K-energy. However, we proved that the converse is also true in [4]. Following suggestion of X. X. Chen, we set out to investigate the relations between

these energy functionals for the general case; in particular, the relations about lower bounds of these functionals.

Now we state our results. Let M be an n-dimensional compact Kähler manifold with $c_1(M) > 0$, and ω be a fixed Kähler metric in the Kähler class $2\pi c_1(M)$. Write

$$\mathcal{P}(M,\omega) = \{\varphi \in C^\infty(M,\mathbb{R}) \mid \omega_\varphi = \omega + \sqrt{-1}\partial\bar\partial\varphi > 0 \text{ on } M\}.$$

For any $k = 0, 1, \cdots, n$, we define the functional $E^0_{k,\omega}(\varphi)$ on $\mathcal{P}(M,\omega)$ by

$$E^0_{k,\omega}(\varphi) = \frac{1}{V}\int_M \Big(\log\frac{\omega_\varphi^n}{\omega^n} - h_\omega\Big)\Big(\sum_{i=0}^k Ric^i_\varphi \wedge \omega^{k-i}\Big) \wedge \omega_\varphi^{n-k} + \frac{1}{V}\int_M h_\omega\Big(\sum_{i=0}^k Ric^i_\omega \wedge \omega^{k-i}\Big) \wedge \omega^{n-k}.$$

Here h_ω is the Ricci potential defined by

$$Ric_\omega - \omega = \sqrt{-1}\partial\bar\partial h_\omega, \quad \text{and} \quad \int_M (e^{h_\omega} - 1)\omega^n = 0.$$

Let $\varphi(t)(t \in [0,1])$ be a path from 0 to φ in $\mathcal{P}(M,\omega)$, we define

$$J_{k,\omega}(\varphi) = -\frac{n-k}{V}\int_0^1 \int_M \frac{\partial\varphi(t)}{\partial t}(\omega^{k+1}_{\varphi(t)} - \omega^{k+1}) \wedge \omega^{n-k-1}_{\varphi(t)} \wedge dt.$$

Then the functional $E_{k,\omega}$ is defined as follows

$$E_{k,\omega}(\varphi) = E^0_{k,\omega}(\varphi) - J_{k,\omega}(\varphi).$$

For simplicity, we will often drop the subscript ω and write E_k instead of $E_{k,\omega}(\varphi)$. The main result of this paper is the following

Theorem 1.1. *For any $k = 1, 2, \cdots, n$, we have*

$$\sum_{i=0}^k (-1)^i \binom{k+1}{i+1} E_{i,\omega}(\varphi) = \frac{1}{V}\int_M u(\sqrt{-1}\partial\bar\partial u)^k \wedge \omega_\varphi^{n-k} + \frac{1}{V}\int_M h_\omega(-\sqrt{-1}\partial\bar\partial h_\omega)^k \wedge \omega^{n-k},$$

where

$$u = \log\frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega.$$

Remark 1.2. *Theorem 1.1 generalizes Pali's formula in [13]. In fact, when $k = 1, 2$, we have the following*

$$\begin{aligned} 2E_0 - E_1 &= -\frac{1}{V}\int_M \sqrt{-1}\partial u \wedge \bar\partial u \wedge \omega_\varphi^{n-1} + c_1, \\ 3E_0 - 3E_1 + E_2 &= -\frac{1}{V}\int_M \sqrt{-1}\partial u \wedge \bar\partial u \wedge \sqrt{-1}\partial\bar\partial u \wedge \omega_\varphi^{n-2} + c_2, \end{aligned}$$

where c_1, c_2 are two constants depending only on ω.

Next we use Theorem 1.1 to get the lower bound of E_k.

Theorem 1.3. *For any positive integer $k = 2, \cdots, n$, and any Kähler metric ω_φ satisfying $Ric_\varphi \geq -\frac{2}{k-1}\omega_\varphi$, we have*

$$E_k(\varphi) \geq (k+1)E_0(\varphi) + c_k,$$

where c_k is a constant defined by

$$c_k = \frac{1}{V}\int_M \sum_{i=0}^{k-1}(-1)^{k-i}\binom{k+1}{i} h_\omega(-\sqrt{-1}\partial\bar{\partial}h_\omega)^{k-i} \wedge \omega^{n-k+i}. \tag{1.1}$$

Remark 1.4. *Theorem 1.3 generalizes some of Song-Weinkove's results in [14]. Since E_0 is bounded from below on $\mathcal{P}(M,\omega)$ on a Kähler-Einstein manifold, from Theorem 1.3 we obtain lower bounds on the functionals E_k under some weaker conditions.*

Remark 1.5. *In [4], we proved that E_1 is bounded from below if and only if E_0 is bounded from below on $\mathcal{P}(M,\omega)$. Using the same method, we also prove that E_0 is bounded from below if and only if the F functional defined by Ding-Tian [8] is bounded from below in [10]. We expect that the lower boundedness of these functionals are equivalent on $\mathcal{P}(M,\omega)$ in [4].*

Finally, we will prove that all the Chen-Tian holomorphic invariants $\mathcal{F}_k$ defined by E_k are the Futaki invariant in the canonical Kähler class.

Theorem 1.6. *For all $k = 0, 1, \cdots, n$, we have*

$$\mathcal{F}_k(X,\omega) = (k+1)\mathcal{F}_0(X,\omega).$$

Remark 1.7. *This result was first proved by C. Liu in [11], and here we give a new proof by using our formula. However, these two methods are essentially the same.*

Acknowledgements: This work was done while I was attending the summmer school on geometric analysis in University of Science and Technology of China (USTC) in 2006, and I would like to express thanks to USTC. I would also like to thank Professor X. X. Chen, W. Y. Ding and X. H. Zhu for their constant support and advice. Thanks also go to Y. Rubinstein, V. Tosatti for pointing out some mistakes in Theorem 1.3, B. Wang , W. Y. He for carefully reading the draft, and the referees for numerous suggestions which helped to improve the presentation.

2 A new formula on E_k

In this section, we will prove Theorem 1.1 and Corollary 2.3.

Proof of Theorem 1.1. By the definition of u, we have

$$\sqrt{-1}\partial\bar{\partial}u = -Ric_\varphi + \omega_\varphi.$$

Therefore, we have

$$\Big(\sum_{p=0}^{i} Ric_\varphi^p \wedge \omega^{i-p}\Big) \wedge \omega_\varphi^{n-i} = \Big(\sum_{p=0}^{i}(\omega_\varphi - \sqrt{-1}\partial\bar{\partial}u)^p \wedge (\omega_\varphi - \sqrt{-1}\partial\bar{\partial}\varphi)^{i-p}\Big) \wedge \omega_\varphi^{n-i}$$

By the definition of E_k^0 we have

$$
\begin{aligned}
&\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}E_i^0(\varphi)\\
=\ & \frac{1}{V}\int_M (u-\varphi)\Big(\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}\sum_{p=0}^{i}(\omega_\varphi-\sqrt{-1}\partial\bar\partial u)^p\wedge(\omega_\varphi-\sqrt{-1}\partial\bar\partial\varphi)^{i-p}\Big)\wedge\omega_\varphi^{n-i}\\
&+\frac{1}{V}\int_M h_\omega\Big(\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}\sum_{p=0}^{i}(\omega+\sqrt{-1}\partial\bar\partial h_\omega)^p\wedge\omega^{i-p}\Big)\wedge\omega^{n-i}.
\end{aligned}
$$

Now we have the following lemma:

Lemma 2.1. *For any two variables x, y and any integer $k>0$, we have*

1.

$$\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}\sum_{p=0}^{i}(1-x)^p(1-y)^{i-p}=\sum_{i=0}^{k}x^{k-i}y^i, \tag{2.2}$$

2.

$$\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}\sum_{p=0}^{i}(1+x)^p=(-x)^k. \tag{2.3}$$

Proof. By direct calculation, we have

$$
\begin{aligned}
&(x-y)\sum_{p=0}^{k}(-1)^p\binom{k+1}{p+1}\sum_{i=0}^{p}(1-x)^i(1-y)^{p-i}\\
=\ & \sum_{p=0}^{k}\binom{k+1}{p+1}((x-1)^{p+1}-(y-1)^{p+1})\\
=\ & x^{k+1}-y^{k+1}.
\end{aligned}
$$

Then the equality (2.2) holds. Similarly, we can prove the equality (2.3). ☐

Thus, the energy functionals E_k^0 satisfy the equality

$$
\begin{aligned}
\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}E_i^0(\varphi) &= \sum_{i=0}^{k}\frac{1}{V}\int_M (u-\varphi)(\sqrt{-1}\partial\bar\partial u)^{k-i}\wedge(\sqrt{-1}\partial\bar\partial\varphi)^i\wedge\omega_\varphi^{n-k}\\
&\quad+\frac{1}{V}\int_M h_\omega(-\sqrt{-1}\partial\bar\partial h_\omega)^k\wedge\omega^{n-k}.
\end{aligned}\tag{2.4}
$$

Observe that for $0\le i\le k-1$,

$$
\begin{aligned}
&\int_M (u-\varphi)(\sqrt{-1}\partial\bar\partial u)^{k-i}\wedge(\sqrt{-1}\partial\bar\partial\varphi)^i\wedge\omega_\varphi^{n-k}\\
=\ & \int_M u\sqrt{-1}\partial\bar\partial(u-\varphi)\wedge(\sqrt{-1}\partial\bar\partial u)^{k-i-1}\wedge(\sqrt{-1}\partial\bar\partial\varphi)^i\wedge\omega_\varphi^{n-k}\\
=\ & \int_M u(\sqrt{-1}\partial\bar\partial u)^{k-i}\wedge(\sqrt{-1}\partial\bar\partial\varphi)^i\wedge\omega_\varphi^{n-k}\\
&-\int_M u(\sqrt{-1}\partial\bar\partial u)^{k-i-1}\wedge(\sqrt{-1}\partial\bar\partial\varphi)^{i+1}\wedge\omega_\varphi^{n-k}.
\end{aligned}
$$

Thus, the equality (2.4) can be written as

$$\begin{aligned}\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}E_i^0(\varphi) &= \frac{1}{V}\int_M u(\sqrt{-1}\partial\bar{\partial}u)^k\wedge\omega_\varphi^{n-k}-\frac{1}{V}\int_M \varphi(\sqrt{-1}\partial\bar{\partial}\varphi)^k\wedge\omega_\varphi^{n-k}\\ &\quad+\frac{1}{V}\int_M h_\omega(-\sqrt{-1}\partial\bar{\partial}h_\omega)^k\wedge\omega^{n-k}. \qquad (2.5)\end{aligned}$$

Next we calculate $J_k(\varphi)$ via a linear path $t\varphi\in\mathcal{P}(M,\omega)$ for $t\in[0,1]$. By the definition of J_k we have

$$\begin{aligned}&\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}J_i(\varphi)\\ =&\ \frac{1}{V}\int_0^1\int_M\sum_{i=0}^{k}-(n-i)(-1)^i\binom{k+1}{i+1}\varphi(\omega_{t\varphi}^{i+1}-(\omega_{t\varphi}-t\sqrt{-1}\partial\bar{\partial}\varphi)^{i+1})\wedge\omega_{t\varphi}^{n-i-1}\wedge dt.\end{aligned}$$

It is easy to check the following lemma:

Lemma 2.2. *Let $B_i=-(n-i)(1-(1-x)^{i+1})$, for any integer $k\geq 1$ we have*

$$\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}B_i=-(n-k)x^{k+1}-(k+1)x^k.$$

Thus, we have

$$\begin{aligned}&\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}J_i(\varphi)\\ =&\ \frac{1}{V}\int_0^1\int_M -(n-k)t^{k+1}\varphi(\sqrt{-1}\partial\bar{\partial}\varphi)^{k+1}\wedge\omega_{t\varphi}^{n-k-1}\wedge dt\\ &-\frac{1}{V}\int_0^1\int_M (k+1)t^k\varphi(\sqrt{-1}\partial\bar{\partial}\varphi)^k\wedge\omega_{t\varphi}^{n-k}\wedge dt\\ =&\ \frac{1}{V}\int_0^1\int_M -\frac{d}{dt}\Big(t^{k+1}\varphi(\sqrt{-1}\partial\bar{\partial}\varphi)^k\wedge\omega_{t\varphi}^{n-k}\Big)\wedge dt\\ =&\ -\frac{1}{V}\int_M\varphi(\sqrt{-1}\partial\bar{\partial}\varphi)^k\wedge\omega_\varphi^{n-k}.\end{aligned}$$

Combining this with the equality (2.5), we have

$$\sum_{i=0}^{k}(-1)^i\binom{k+1}{i+1}E_i(\varphi)=\frac{1}{V}\int_M u(\sqrt{-1}\partial\bar{\partial}u)^k\wedge\omega_\varphi^{n-k}+\frac{1}{V}\int_M h_\omega(-\sqrt{-1}\partial\bar{\partial}h_\omega)^k\wedge\omega^{n-k}.$$

□

Next we will use Theorem 1.1 to prove the following corollary.

Corollary 2.3. *Let*

$$F_k(\varphi)=\frac{1}{V}\int_M u(\sqrt{-1}\partial\bar{\partial}u)^k\wedge\omega_\varphi^{n-k}+\frac{1}{V}\int_M h_\omega(-\sqrt{-1}\partial\bar{\partial}h_\omega)^k\wedge\omega^{n-k},$$

we have

1. *For nonnegative integers p, k $(0 \le p \le k-2 \le n-2)$, we have*

$$\sum_{i=p}^{k} (-1)^i \binom{k-p}{i-p} E_i = \sum_{i=0}^{p+1} (-1)^i \binom{p+1}{i} F_{k-i}. \tag{2.6}$$

2. *For any positive integer $k = 1, 2, \cdots, n$, we have*

$$E_k - E_{k-1} - E_0 = \frac{1}{V}\int_M u\Big(Ric_{\varphi}^k - \omega_{\varphi}^k\Big) \wedge \omega_{\varphi}^{n-k} + \frac{1}{V}\int_M h_{\omega}\Big(Ric_{\omega}^k - \omega^k\Big) \wedge \omega^{n-k}. \tag{2.7}$$

3. *For any positive integer $k = 1, 2, \cdots, n$, we have*

$$E_k = \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k+1}{i} F_{k-i} + (k+1)E_0. \tag{2.8}$$

Proof. (1). We show this by induction on p. The corollary holds for $p = 0$. In fact, by Theorem 1.1 we have

$$\sum_{i=0}^{k} (-1)^i \binom{k+1}{i+1} E_i = F_k, \tag{2.9}$$

$$\sum_{i=0}^{k-1} (-1)^i \binom{k}{i+1} E_i = F_{k-1}. \tag{2.10}$$

Subtract (2.10) from (2.9), we have

$$\sum_{i=p}^{k} (-1)^i \binom{k}{i} E_i = F_k - F_{k-1}.$$

We assume that the corollary holds for p, then

$$\sum_{i=p}^{k} (-1)^i \binom{k-p}{i-p} E_i = \sum_{i=0}^{p+1} (-1)^i \binom{p+1}{i} F_{k-i}, \tag{2.11}$$

$$\sum_{i=p}^{k-1} (-1)^i \binom{k-p-1}{i-p} E_i = \sum_{i=0}^{p+1} (-1)^i \binom{p+1}{i} F_{k-i-1} = \sum_{i=1}^{p+2} (-1)^{i-1} \binom{p+1}{i-1} F_{k-i}. \tag{2.12}$$

Subtract (2.12) from (2.11), we have

$$\begin{aligned}
\sum_{i=p+1}^{k} (-1)^i \binom{k-p-1}{i-p-1} E_i &= \sum_{i=0}^{p+1} (-1)^i \binom{p+1}{i} F_{k-i} - \sum_{i=1}^{p+2} (-1)^{i-1} \binom{p+1}{i-1} F_{k-i} \\
&= F_k + \sum_{i=1}^{p+1} (-1)^i \left(\binom{p+1}{i} + \binom{p+1}{i-1}\right) F_{k-i} + (-1)^{p+2} F_{k-p-2} \\
&= \sum_{i=0}^{p+2} (-1)^i \binom{p+2}{i} F_{k-i}.
\end{aligned}$$

The corollary holds for $p+1$. Thus, the equality (2.6) holds.

(2) We can show the following formula by induction:

$$E_k - E_{k-1} = \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k}{i} F_{k-i} + E_0. \tag{2.13}$$

In fact, by Theorem 1.1 the formula (2.13) holds for $k=1$. We assume the formula (2.13) holds for some integer $k \le n-1$, then by (1) we have

$$E_{k+1} = 2E_k - E_{k-1} + \sum_{i=0}^{k} (-1)^{i-k-1} \binom{k}{i} F_{k+1-i}. \tag{2.14}$$

Thus, we have

$$\begin{aligned} E_{k+1} - E_k &= E_k - E_{k-1} + \sum_{i=0}^{k} (-1)^{i-k-1} \binom{k}{i} F_{k+1-i} \\ &= \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k}{i} F_{k-i} + E_0 + \sum_{i=0}^{k} (-1)^{i-k-1} \binom{k}{i} F_{k+1-i} \\ &= E_0 + \sum_{i=0}^{k} (-1)^{k+1-i} \binom{k+1}{i} F_{k+1-i}. \end{aligned}$$

Then the formula (2.13) holds for $k+1$.

On the other hand, by direct calculation we have

$$\begin{aligned} \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k}{i} F_{k-i} &= \frac{1}{V}\int_M u\Big(\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} (\sqrt{-1}\partial\bar\partial u)^{k-i} \wedge \omega_\varphi^i - \omega_\varphi^k\Big) \wedge \omega_\varphi^{n-k} \\ &\quad + \frac{1}{V}\int_M h_\omega\Big(\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} (-\sqrt{-1}\partial\bar\partial h_\omega)^{k-i} \wedge \omega^i - \omega^k\Big) \wedge \omega^{n-k} \\ &= \frac{1}{V}\int_M u\Big((\omega_\varphi - \sqrt{-1}\partial\bar\partial u)^k - \omega_\varphi^k\Big) \wedge \omega_\varphi^{n-k} + \frac{1}{V}\int_M h_\omega\Big(Ric_\omega^k - \omega^k\Big) \wedge \omega^{n-k} \\ &= \frac{1}{V}\int_M u\Big(Ric_\varphi^k - \omega_\varphi^k\Big) \wedge \omega_\varphi^{n-k} + \frac{1}{V}\int_M h_\omega\Big(Ric_\omega^k - \omega^k\Big) \wedge \omega^{n-k}. \end{aligned}$$

Then the equality (2.7) holds.

(3). We prove this result by induction on k. The corollary holds for $k=1$ obviously. We assume that it holds for integers less than k, then by (1) we have

$$E_k = 2E_{k-1} - E_{k-2} + \sum_{i=0}^{k-1} (-1)^{i-k} \binom{k-1}{i} F_{k-i}.$$

By induction, we have

$$E_{k-1} = \sum_{i=0}^{k-2} (-1)^{k-i-1} \binom{k}{i} F_{k-i-1} + kE_0 = \sum_{i=1}^{k-1} (-1)^{k-i} \binom{k}{i-1} F_{k-i} + kE_0,$$

and

$$E_{k-2}=\sum_{i=0}^{k-3}(-1)^{k-i-2}\binom{k-1}{i}F_{k-i-2}+(k-1)E_0=\sum_{i=2}^{k-1}(-1)^{k-i}\binom{k-1}{i-2}F_{k-i}+(k-1)E_0.$$

Then we have

$$\begin{aligned}E_k &= 2\Big(\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k}{i-1}F_{k-i}+kE_0\Big)-\Big(\sum_{i=2}^{k-1}(-1)^{k-i}\binom{k-1}{i-2}F_{k-i}+(k-1)E_0\Big)\\ &\quad+\sum_{i=0}^{k-1}(-1)^{i-k}\binom{k-1}{i}F_{k-i}\\ &= \sum_{i=0}^{k-1}(-1)^{k-i}\binom{k+1}{i}F_{k-i}+(k+1)E_0.\end{aligned}$$

Then the equality (2.8) holds.

□

3 Applications of the new formula

In this section, we will prove Theorem 1.3 and 1.6.

3.1 On the lower bound of E_k

Proof of Theorem 1.3. By the equality (2.8) of Corollary 2.3, we have

$$\begin{aligned}E_k-(k+1)E_0 &= \sum_{i=0}^{k-1}(-1)^{k-i}\binom{k+1}{i}F_{k-i}\\ &= \frac{1}{V}\int_M\sum_{i=0}^{k-1}(-1)^{k-i}\binom{k+1}{i}u(\sqrt{-1}\partial\bar\partial u)^{k-i}\wedge\omega_\varphi^{n-k+i}+c_k\\ &= \frac{1}{V}\int_M\sqrt{-1}\partial u\wedge\bar\partial u\wedge\Big(\sum_{i=0}^{k-1}(-1)^{k-i-1}\binom{k+1}{i}(\sqrt{-1}\partial\bar\partial u)^{k-i-1}\wedge\omega_\varphi^i\Big)\wedge\omega_\varphi^{n-k}+c_k\\ &= \frac{1}{V}\int_M\sqrt{-1}\partial u\wedge\bar\partial u\wedge\Big(\sum_{i=0}^{k-1}\binom{k+1}{i}(Ric_\varphi-\omega_\varphi)^{k-i-1}\wedge\omega_\varphi^i\Big)\wedge\omega_\varphi^{n-k}+c_k,\end{aligned}$$

where c_k is a constant defined by (1.1). Observe that

$$\sum_{i=0}^{k-1}\binom{k+1}{i}(Ric_\varphi-\omega_\varphi)^{k-i-1}\wedge\omega_\varphi^i=\sum_{i=1}^{k}iRic_\varphi^{k-i}\wedge\omega_\varphi^{i-1}. \tag{3.15}$$

Then we need to check when (3.15) is nonnegative. Obviously, this is true when $Ric_\varphi\geq 0$, Here we want to get a better condition on Ricci curvature. If $k=2$, we need to assume $Ric_\varphi\geq -2\omega_\varphi$. Now we assume $k\geq 3$. Set

$$P(x)=\sum_{i=1}^{k}ix^{k-i}=(x+\frac{2}{k-1})^{k-1}+\sum_{i=2}^{k}a_i(x+\frac{2}{k-1})^{k-i},$$

where a_i are the constants defined by

$$a_i = \frac{1}{(k-i)!} P^{(k-i)}(-\frac{2}{k-1}).$$

By Lemma A.1 in the appendix, $a_i \geq 0$. Then if $Ric_\varphi \geq -\frac{2}{k-1}\omega_\varphi$, we have

$$\sum_{i=1}^{k} iRic_\varphi^{k-i} \wedge \omega_\varphi^{i-1} = \Big(Ric_\varphi + \frac{2}{k-1}\omega_\varphi\Big)^{k-1} + \sum_{i=2}^{k} a_i \Big(Ric_\varphi + \frac{2}{k-1}\omega_\varphi\Big)^{k-i} \wedge \omega_\varphi^{i-1} \geq 0.$$

Therefore, $E_k \geq (k+1)E_0 + c_k$. □

3.2 On the holomorphic invariants $\mathcal{F}_k$

In this subsection, we will use the equality (2.8) of Corollary 2.3 to prove that all the holomorphic invariants defined in [5] are the Futaki invariant. This result was first obtained by Liu in [11]. Here we give a new proof by using our formula.

Let X be a holomorphic vector field. Then by $c_1(M) > 0$, we can decompose $i_X\omega$ as $i_X\omega = \sqrt{-1}\bar{\partial}\theta_X$, where θ_X is a potential function of X with respect to ω.

Definition 3.1. *(cf. [5]) For any holomorphic vector field X, we define*

$$\mathcal{F}_k = (n-k)\int_M \theta_X \omega^n + \int_M \Big((k+1)\Delta\theta_X Ric_\omega^k \wedge \omega^{n-k} - (n-k)\theta_X Ric_\omega^{k+1} \wedge \omega^{n-k-1}\Big).$$

It was proved that $\mathcal{F}_k$ is a holomorphic invariant. When $k = 0$, we have

$$\mathcal{F}_0(X, \omega) = n\int_M X(h_\omega)\omega^n,$$

which is a multiple of the Futaki invariant.

Proposition 3.2. *(cf. [5]) Let $\{\Phi(t)\}_{|t|<\infty}$ be the one-parameter subgroup of automorphisms induced by $Re(X)$. Then*

$$\frac{dE_k(\varphi_t)}{dt} = \frac{1}{V} Re(\mathcal{F}_k(X, \omega)),$$

where φ_t are the Kähler potentials of $\Phi_t^*\omega$, i.e.,$\Phi_t^*\omega = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$.

Now we can prove Theorem 1.6.

Proof of Theorem 1.6. By Corollary 2.3, we only need to show

$$\frac{dF_k(\varphi_t)}{dt} = 0,$$

for all k, where φ_t is the Kähler potential defined in the previous proposition. Differentiating $\omega_\varphi = \Phi_t^*\omega = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$, we get

$$L_{Re(X)}\omega_\varphi = \sqrt{-1}\partial\bar{\partial}\frac{\partial\varphi_t}{\partial t}.$$

On the other hand, since $L_X\omega = \sqrt{-1}\partial\bar\partial\theta_X$, we have

$$\frac{\partial\varphi_t}{\partial t} = Re(\theta_X(\varphi)) + c,$$

where c is a constant and $\theta_X(\varphi) = \theta_X + X(\varphi)$. By the definition of u, we have

$$Ric_\varphi - \omega_\varphi = -\sqrt{-1}\partial\bar\partial u.$$

Take the inner product on both sides, we have

$$-\Delta\theta_X(\varphi) - \theta_X(\varphi) = -X(u).$$

Here Δ is the Lapacian with respect to ω_φ. On the other hand

$$\frac{\partial u}{\partial t} = \Delta\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial t} = Re(\Delta\theta_X(\varphi) + \theta_X(\varphi)) + c = Re(X(u)) + c.$$

Thus,

$$\begin{aligned}
& \frac{\partial}{\partial t}\int_M u(\sqrt{-1}\partial\bar\partial u)^k \wedge \omega_\varphi^{n-k} \\
= & \int_M \frac{\partial u}{\partial t}(\sqrt{-1}\partial\bar\partial u)^k \wedge \omega_\varphi^{n-k} + \int_M ku\sqrt{-1}\partial\bar\partial\frac{\partial u}{\partial t} \wedge (\sqrt{-1}\partial\bar\partial u)^{k-1} \wedge \omega_\varphi^{n-k} \\
& + \int_M (n-k)u(\sqrt{-1}\partial\bar\partial u)^k \wedge \sqrt{-1}\partial\bar\partial\frac{\partial\varphi}{\partial t} \wedge \omega_\varphi^{n-k-1} \\
= & Re\Big(\int_M (k+1)X(u)(\sqrt{-1}\partial\bar\partial u)^k \wedge \omega_\varphi^{n-k} + (n-k)\theta_X(\varphi)(\sqrt{-1}\partial\bar\partial u)^{k+1} \wedge \omega_\varphi^{n-k-1}\Big) \\
= & Re\Big(\int_M i_X(\partial u(\sqrt{-1}\partial\bar\partial u)^k \wedge \omega_\varphi^{n-k})\Big) \\
= & 0.
\end{aligned}$$

Thus, by the equality (2.8) in Corollary 2.3, we have

$$\frac{dE_k(\varphi_t)}{dt} = \sum_{i=0}^{k-1}(-1)^{k-i}\binom{k+1}{i}\frac{d}{dt}F_{k-i}(\varphi_t) + (k+1)\frac{dE_0(\varphi_t)}{dt} = \frac{k+1}{V}Re(\mathcal{F}_0(X,\omega)).$$

By Proposition 3.2, the theorem is proved. □

A An elementary lemma

In the proof of Theorem 1.3, we need to use the following lemma.

Lemma A.1. *Let m be a positive integer. Consider the polynomial*

$$P(x) = x^m + 2x^{m-1} + \cdots + mx + (m+1),$$

then for any $i (0 \le i \le m)$, the i^{th} derivative of the polynomial at the point $x = -\frac{2}{m}$ is nonnegative.

Proof. The i^{th} derivative of the polynomial is

$$P^{(i)}(x) = \sum_{p=0}^{m-i} (m-i+1-p)(i+p)(i+p-1)\cdots(p+1)x^p.$$

For simplicity, we define $a(p,i)$ by

$$a(p,i) = (i+p)(i+p-1)\cdots(p+1).$$

If $m-i$ is even, then

$$P^{(i)}(x) = a(m-i,i)x^{m-i} + \sum_{p=0}^{\frac{m-i}{2}-1} \Big((m-i+1-2p)\ a(2p,i)x^{2p} + (m-i-2p)a(2p+1,i)x^{2p+1}\Big). \tag{1.16}$$

If $m-i$ is odd, we write $P^{(i)}(x)$ as

$$P^{(i)}(x) = \sum_{p=0}^{\frac{m-i-1}{2}} \Big((m-i+1-2p)\ a(2p,i)x^{2p} + (m-i-2p)a(2p+1,i)x^{2p+1}\Big). \tag{1.17}$$

Note that $P^{(m-1)}(-\frac{2}{m}) = 0$, so we can assume $i \leq m-2$. Since the lemma is trivial for $1 \leq m \leq 10$, we assume $m > 10$. For simplicity, we define

$$A_p(x) = (m-i+1-2p)\ a(2p,i)x^{2p} + (m-i-2p)a(2p+1,i)x^{2p+1}.$$

Claim A.2. *If $1 \leq p \leq \frac{m-i-1}{2}$, we have $A_p(-\frac{2}{m}) > 0$.*

Proof. We need to show

$$\frac{(m-i+1-2p)}{m-i-2p}\frac{m(2p+1)}{2(i+2p+1)} > 1.$$

Since $1 \leq p \leq \frac{m-i-1}{2}$, this is obvious because

$$\frac{m(2p+1)}{2(i+2p+1)} \geq \frac{3m}{2m} > 1.$$

The claim is proved. □

By Claim A.2, all the terms on the right hand side of (1.16) and (1.17) are positive except $A_0(-\frac{2}{m})$. Note that if $0 \leq i \leq \frac{m}{2}$,

$$\begin{aligned} A_0(-\frac{2}{m}) &= (m-i)(i+1)\cdots 2(-\frac{2}{m}) + (m-i+1)i(i-1)\cdots 1 \\ &= \frac{i!}{m}(m-2i)(m-i-1) \\ &\geq 0. \end{aligned}$$

So it only remains to deal with the case $i > \frac{m}{2}$. Now, we consider the case $\frac{1}{2}m < i \leq m-5$. The following claim shows that $A_0 + A_1 + A_2$ is positive at $x = -\frac{2}{m}$ in this case.

Claim A.3. *If $\frac{1}{2}m < i \leq m-5$, then $(A_0 + A_1 + A_2)(-\frac{2}{m}) > 0$.*

Proof. In fact,

$$\begin{aligned} & \frac{120}{i!}(A_0 + A_1 + A_2)(-\frac{2}{m}) \\ = \; & -\frac{32}{m^5}(m-i-4)(i+5)(i+4)(i+3)(i+2)(i+1) + \frac{80}{m^4}(m-i-3)(i+4)(i+3)(i+2)(i+1) \\ & -\frac{160}{m^3}(m-i-2)(i+3)(i+2)(i+1) + \frac{240}{m^2}(m-i-1)(i+2)(i+1) \\ & -\frac{240}{m}(m-i)(i+1) + 120(m-i+1). \end{aligned}$$

Observe that

$$\frac{32}{m^4}(m-i-3)(i+4)(i+3)(i+2)(i+1) > \frac{32}{m^5}(m-i-4)(i+5)(i+4)(i+3)(i+2)(i+1),$$

so we only need to show

$$\begin{aligned} A \;&:=\; \frac{48}{m^4}(m-i-3)(i+4)(i+3)(i+2)(i+1) - \frac{160}{m^3}(m-i-2)(i+3)(i+2)(i+1) \\ & \quad + \frac{240}{m^2}(m-i-1)(i+2)(i+1) - \frac{240}{m}(m-i)(i+1) + 120(m-i+1) > 0. \end{aligned}$$

Let $y = \frac{i+5}{m} \in (0.5, 1]$ and $\epsilon = \frac{1}{m}$. Then

$$\begin{aligned} \frac{A}{8m} &= 6(1-y+2\epsilon)(y-\epsilon)(y-2\epsilon)(y-3\epsilon)(y-4\epsilon) - 20(1-y+3\epsilon)(y-2\epsilon)(y-3\epsilon)(y-4\epsilon) \\ & \quad +30(1-y+4\epsilon)(y-3\epsilon)(y-4\epsilon) - 30(1-y+5\epsilon)(y-4\epsilon) + 15(1-y+6\epsilon) \\ &= 288\epsilon^5 + (1584 - 744y)\epsilon^4 + (720y^2 - 2340y + 1920)\epsilon^3 + (960 + 1270y^2 - 330y^3 - 1720y)\epsilon^2 \\ & \quad +(210 + 72y^4 - 480y + 510y^2 - 300y^3)\epsilon + 15 - 45y - 50y^3 + 60y^2 + 26y^4 - 6y^5. \end{aligned}$$

We can check that all these coefficients of ϵ are nonnegative for $y \in (0.5, 1]$, so $A > 0$ and the claim is proved. ◻

Remark A.4. *The sum of the last four terms $(A_0 + A_1)(-\frac{2}{m})$ may be negative when $\frac{m}{2} < i \leq m-2$. In fact, if $i = \frac{9}{10}m$ and m is sufficiently large, then*

$$\begin{aligned} \frac{6}{i!}(A_0 + A_1)(-\frac{2}{m}) &= -(m-i-2)(i+3)(i+2)(i+1)\frac{8}{m^3} + (m-i-1)(i+2)(i+1)\frac{12}{m^2} \\ & \quad -(m-i)(i+1)\frac{12}{m} + 6(m-i+1) \\ &\sim -0.0912m < 0. \end{aligned}$$

Next we consider the case $m-4 \leq i \leq m-2$.

Claim A.5. *The lemma holds for $m-4 \leq i \leq m-2$.*

Proof. The proof is easy. If $i = m-4$, then

$$\begin{aligned}
\frac{6}{i!}P^{(m-4)}(-\frac{2}{m}) &\geq 30 - \frac{48(m-3)}{m} + \frac{36}{m^2}(m-2)(m-3) - \frac{16}{m^3}(m-1)(m-2)(m-3) \\
&\geq 30 - \frac{48(m-3)}{m} + \frac{20}{m^2}(m-2)(m-3) \\
&= \frac{2m^2+44m+120}{m^2} > 0.
\end{aligned}$$

Similarly, we can prove that the lemma holds for $i = m-3, m-2$. □

By Claim A.2-A.5, the lemma is proved.

□

References

[1] S. Bando, T. Mabuchi. Uniqueness of Einstein Kähler metrics modulo connected group actions. Algebraic geometry, Sendai, 1985, 11–40, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[2] H. D. Cao, G. Tian, X. H. Zhu. Kähler Ricci solitions on compact complex manifolds with $c_1(M) > 0$, Geom. funct. anal., 15(2005), 697-719.

[3] X. X. Chen. On the lower bound of energy functional $E_1(I)$– a stability theorem on the Kähler Ricci flow. J. Geometric Analysis. 16 (2006) 23-38.

[4] X. X. Chen, H. Li, B. Wang. On the lower bound of energy functional $E_1(II)$. math.DG/0609694.

[5] X. X. Chen, G. Tian. Ricci flow on Kähler-Einstein surfaces. Invent. Math. 147 (2002), no. 3, 487–544.

[6] X. X. Chen, G. Tian. Ricci flow on Kähler-Einstein manifolds. Duke. Math. J. 131, (2006), no. 1, 17-73.

[7] X. X. Chen, G. Tian. Geometry of Kähler Metrics and Foliations by Holomorphic Discs. math.DG/0507148.

[8] W. Y. Ding, G. Tian. The generalized Moser-Trudinger inequality. Proceedings of Nankai International Conference of Nonlinear Analysis, 1993.

[9] S. K. Donaldson. Scalar curvature and stability of toric varieties. J. Differential Geom. 62 (2002), no. 2, 289–349.

[10] H. Li. On the lower bound of the K energy and F functional. math.DG/0609725.

[11] C. J. Liu. Bando-Futaki Invariants on Hypersurfaces. math.DG/0406029.

[12] T. Mabuchi. K-energy maps integrating Futaki invariants. Tohoku Math. J. (2) 38(1986), no. 4, 575-593.

[13] N. Pali. A consequence of a lower bound of the K-energy. Int. Math. Res. Not. 2005, no. 50, 3081–3090.

[14] J. Song, B. Weinkove. Energy functionals and canonical Kahler metrics. math.DG/0505476.

[15] G. Tian. On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$. Invent. Math. 89 (1987), no. 2, 225–246.

[16] G. Tian. Kähler-Einstein metrics with positive scalar curvature. Invent. Math. 130 (1997), no. 1, 1–37.

[17] G. Tian. Canonical metrics in Kähler geometry. Notes taken by Meike Akveld. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2000.

[18] G. Tian, X. H. Zhu, A new holomorphic invariant and uniqueness of Kähler-Ricci solitons. Comment. Math. Helv. 77(2002), 297-325.

[19] V. Tosatti. On the Critical Points of the E_k Functionals in Kähler Geometry. math.DG/0506021.

School of Mathematical Sciences, Peking University, Beijing, 100871, P.R. China
Email: lihaozhao@gmail.com